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HIGH-FREQUENCY NATURAL OSCILLATIONS OF MECHANICAL SYSTEMS†

L. D. AKULENKO

Moscow

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Higher modes of natural oscillations of a wide class of mechanical systems, described by general boundary-value problems with various types of boundary conditions, are investigated. An effective method of determining the oscillation frequencies and shapes, based on the use of asymptotic methods of non-linear mechanics (the averaging, accelerated convergence and asymptotic expansion methods) is developed. Expressions for the eigenvalues (frequencies) and eigenfunctions (shapes) are obtained in an explicit form with the required degree of accuracy in negative powers of the order number of the mode, and a justification of the method is given. The eigenvalues for specific mechanical systems, which perform free or parametric oscillations, are calculated. The oscillations of a homogeneous rod and the transverse vibrations of a tightly stretched inhomogeneous string are considered. The higher resonance zones in Hill's problem of parametric oscillations and in the problem of small spatial oscillations of a dynamically symmetrical satellite whose polar axis performs non-linear oscillations in the plane of a circular orbit are investigated. Some mechanical effects are detected and described. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will investigate the natural oscillations of mechanical systems described by a generalized boundary-value problem for eigenvalues and eigenfunctions of the form [1–4]

$$\begin{gathered}\left(p(x,\lambda)u'\right)' + f(x,\lambda)u = 0, \quad u(0) = u(L) = 0 \\ 0 < c_1 \leqslant p, f \leqslant c_2 < \infty, \quad 0 \leqslant x \leqslant L < \infty, \quad \lambda \in \Lambda\end{gathered} \tag{1.1}$$

Here x is the argument (the Euler linear coordinate or time) and L is a specified quantity. The coefficients p and f are determined by the mechanical nature of problem (1.1) (see Section 6). These functions are assumed to be sufficiently continuous in the region of variation of the variables x and λ considered and possess certain structural properties. The parameter λ represents the oscillation frequency or a physical parameter of the system, henceforth we will investigate the situation where its values can be as large as desired ($\lambda \to \infty$). It is assumed that relations (1.1) can be reduced to dimensionless variables. The problem arises of determining the real eigenvalues λ_n and eigenfunctions $u_n(x)$ for sufficiently large $n \gg 1$ [5–7].

In problem (1.1), to fix our ideas, we will consider the boundary conditions of rigidly clamped ends [1, 5]. If necessary, one can take more general flexible clamping conditions or conditions in which one or both ends are free (see Section 5). Mixed boundary conditions or conditions of periodicity are also of some practical interest [1, 2, 5]. Note that the differential operator in problem (1.1) belongs to a special case of a wide class of non-self-conjugate linear operators [2, 3, 7].

The main results have been obtained for the self-conjugate boundary-value problem with $f = \lambda r(x) - q_0(x)$; it is usually assumed that $p = r = 1$. We have developed a theory, and also analytical and numerical methods of constructing the solution $\lambda_n, u_n(x)$ (variational methods, the finite-elements method, the accelerated convergence method etc. [1, 2, 5–8]. Numerical methods are effective for relatively small values of $n \sim 1$, i.e. for lower modes of oscillation. An investigation of the behaviour of the solution when $n \gg 1$, and, in particular, the construction of the asymptotic form as $n \to \infty$ is also important from theoretical and applied points of view. Asymptotic representations for $\lambda_n, u_n(x)$ were obtained constructively in the leading term of the expansion [5–7]. More accurate approximate expressions, which given an absolute error as small as desired and are suitable for practical calculations for comparatively small values of $n \sim 1$ also, will be useful in practice.

It is proposed to develop an effective numerical-analytical method of constructing a highly accurate approximate solution of generalized boundary-value problem (1.1) when certain requirements imposed on the structure of the

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functions p and f are satisfied, namely

$$p \equiv p(x), \quad f \equiv \omega^2 r(x) - \omega q(x,\varepsilon), \quad \lambda = \omega^2, \quad \varepsilon = \omega^{-1}$$
$$0 < c_1 \leqslant p, r \leqslant c_2 < \infty, \quad 0 < \varepsilon \leqslant \varepsilon_0, \quad \varepsilon_0 \ll 1, \quad \omega \gg 1 \tag{1.2}$$

Functions of the form (1.2) are often encountered in practical applications (see, for example, section 6.4). The parameter ω characterizes the frequency while ε is the oscillation period, it is required to determine the eigenvalues of these quantities. The functions r and q are assumed to be continuous in x and, moreover, q depends on ε in a regular manner: it is continuous and satisfies the Lipschitz condition. Note that, by means of the non-singular linear change of variables $u^* = u\sqrt{p}$, Eq. (1.1) can be reduced to the form in which $p \equiv 1$ [2, 5]. The function $p(x)$ must be sufficiently smooth and not change in sign (see (1.2)).

A special case of (1.2) where $q \equiv \varepsilon q_0(x)$, i. e. $q(x, 0) \equiv 0$, is investigated in the classical Sturm–Liouville problem. In this case the following fundamental result is obtained [5]

$$\lambda_n = (\pi n)^2 \left(\int_0^L \nu_0(x)dx \right)^{-2} + O(1), \quad u_n(x) = \left(\frac{2}{L}\right)^{1/2} (\mu_0(x))^{1/2} \sin\left(\frac{\pi n}{L} \int_0^x \nu_0(y)dy \right) + O\left(\frac{1}{n}\right)$$
$$\nu_0(x) = (r(x)/p(x))^{1/2}, \quad \mu_0(x) = (r(x)p(x))^{-1/2}, \quad (u_n, u_m)_r = \delta_{nm} + O\left(n^{-1}, m^{-1}\right) \tag{1.3}$$

In approximation (1.3) the function $q_0(x)$ is immaterial, which confirms the need to refine these expressions.

The problem of highly accurate asymptotic expansions was discussed in [6, 7], but the formulae derived have no constructive character since they are not represented in terms of the initial coefficients; the results obtained have an implicit form and require, as a rule, the solutions of transcendental equations.

We will describe an effective method for constructing an approximate solution of generalized boundary-value problem (1.1), (1.2), based on the methods of non-linear mechanics (averaging, accelerated convergence and asymptotic expansions [9–12]). It allows of natural extensions to wider classes of boundary-value problems (see below). Note that the problem in question corresponds to relative asymptotic closeness of the functions $f(x, \lambda)$ and $\lambda r(x) - q_0(x)$ as $\lambda \to \infty$; the absolute values may differ considerably (without limit) (by an amount $O(\sqrt{\lambda})$).

2. THE USE OF ASYMPTOTIC METHODS OF NON-LINEAR MECHANICS

Consider the generalized boundary-value problem (1.1), (1.2) for asymptotically large values of the parameter λ. We will assume that these relations can be represented in dimensionless form, i. e. λ, ω, ε and L are numerical parameters, where $\lambda, \omega \gg 1$, $\varepsilon \ll 1$, $L \sim 1$. We will introduce the argument $s = \omega x$, which varies in the asymptotically large interval $0 \leqslant x \leqslant L\varepsilon^{-1}$. Then, (1.1) can be reduced to the form of the equation of an oscillatory system with slowly varying parameters: $x = \varepsilon s$.

We will represent the boundary-value problem in the standard "amplitude-phase" a, φ variables [9–11]

$$da/ds = \varepsilon a F_+(x, \varphi, \varepsilon), \quad u = a\sin\varphi, \quad du/ds = a\nu_0 \cos\varphi, \quad a > 0$$
$$d\varphi/ds = \nu_0(x) + \varepsilon N_+(x, \varphi, \varepsilon), \quad \varphi(0) = 0, \quad \varphi\left(L\varepsilon^{-1}\right) = \pi n$$
$$F_\pm \equiv \frac{1}{2} g(x,\varepsilon)\sin 2\varphi - \frac{1}{2} h(x)(\pm 1 + \cos 2\varphi), \tag{2.1}$$
$$N_\pm \equiv -\frac{1}{2} g(x,\varepsilon)(1 \mp \cos 2\varphi) + \frac{1}{2} h(x)\sin 2\varphi$$
$$g \equiv q(x,\varepsilon)\mu_0(x), \quad h \equiv -(\ln \mu_0(x))', \quad 0 < r_1/p_2 \leqslant \nu_0 \leqslant r_2/p_1 < \infty$$

The derivative with respect to the "fast" argument $s = \omega x$ characterizes the rate of variation of the phase φ: $d\varphi/ds \approx \nu_0 \sim 1$, and x is the "slow" variable. Note that the right-hand sides of Eqs (2.1) are π-periodic in φ and, moreover, the variable φ is separated. We have for it the boundary-value problem in the

asymptotically large interval $0 \leqslant s \leqslant L\varepsilon^{-1}$, the value of which is determined by the value of $n \gg 1$. The parameter $\varepsilon = \omega^{-1}$ is not specified and must be found from the boundary condition. After calculating the function $\varphi(s, \varepsilon)$ and the quantities $\varepsilon_n = \varepsilon(n)$, the unknown $a_n = a(s, \varepsilon_n)$ is found by a simple quadrature (see Section 4).

$$\varphi = \psi + \varepsilon V(x,\psi,\varepsilon) = \psi + \varepsilon V_1(x,\psi,\varepsilon) + \varepsilon^2 V_2 + \ldots + \varepsilon^k V_k + \varepsilon^{k+1} \ldots$$
$$d\psi / ds = \nu_0(x) + \varepsilon M(x,\varepsilon) = \nu_0(x) + \varepsilon\nu_1(x,\varepsilon) + \varepsilon^2\nu_2 + \ldots + \varepsilon^k\nu_k + \varepsilon^{k+1} \ldots \tag{2.2}$$

The required fast phase $\varphi(s, \varepsilon)$ for $0 \leqslant s \leqslant L\varepsilon^{-1}$ is constructed approximately using the standard scheme of the averaging method. It involves determining the change of variables $\varphi \to \psi$, close to identity, and such that the equation for the new "averaged" phase ψ has a definite structure with the required degree of accuracy in ε [9–11].

Here V_i and ν_i are functions to be determined, which depend regularly on ε via $g(x, \varepsilon)$. The degree of accuracy k of expansions (2.2) is limited solely by the smoothness of the function N_+ with respect to x, since it is analytical with respect to φ. If terms $O(\varepsilon^k)$ inclusive are taken into account we obtain an error $O(\varepsilon^k)$ in the interval $0 \leqslant s \leqslant L\varepsilon^{-1}$. The unknown functions V_i and ν_i are found in a standard way taking into account the condition of π-periodicity with respect to ψ [9–14]. For the averaged derivative $d\psi/ds$ we have the explicit analytic representations

$$\frac{d\psi}{ds} = \nu^{(k)}(x,\varepsilon) + \varepsilon^{k+1} \ldots, \quad \nu^{(k)}(x,\varepsilon) \equiv \sum_{i=0}^{k} \varepsilon^i \nu_i(x,\varepsilon)$$
$$\nu_0(x) = \left(\frac{r(x)}{p(x)}\right)^{1/2}, \quad \nu_1(x,\varepsilon) = \langle N_+ \rangle = -\frac{1}{2} g(x,\varepsilon)$$
$$\nu_2 = -\left\langle \frac{\partial V_1}{\partial x} \right\rangle + \left\langle \frac{\partial N_+}{\partial \psi} V_1 \right\rangle = -\frac{1}{4} H' - \frac{\nu_0}{8}\left(H^2 + G^2\right), \quad H \equiv \frac{h}{\nu_0} \tag{2.3}$$
$$\nu_3 = -\left\langle \frac{\partial V_2}{\partial x} \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 N_+}{\partial \psi^2} V_1^2 \right\rangle + \left\langle \frac{\partial N_+}{\partial \psi} V_2 \right\rangle =$$
$$= -\frac{1}{32} HG' - \frac{3}{32} H'G - \frac{\nu_0^2}{32} G\left(H^2 + G^2\right) + \frac{1}{8}\left(\frac{G'}{\nu_0}\right)', \quad G \equiv \frac{g}{\nu_0}$$

etc. The angle brackets in (2.3) and henceforth denote averaging over ψ of the corresponding π-periodic functions, which have an elementary trigonometric structure, which enables the mean values to be calculated in an explicit form. The functions $V_i(x, \psi, \varepsilon)$ are also defined explicitly

$$V_1(x,\psi,\varepsilon) = \frac{1}{\nu_0} \int_0^{\psi} \left(N_+ - \langle N_+ \rangle\right) d\varphi = \frac{1}{2\nu_0(x)} F_-(x,\psi,\varepsilon)$$
$$V_2(x,\psi,\varepsilon) = \frac{1}{\nu_0} \int_0^{\psi} \left(\left\langle \frac{\partial V_1}{\partial x} \right\rangle - \frac{\partial V_1}{\partial x} + \frac{\partial N_+}{\partial \psi} V_1 - \left\langle \frac{\partial N_+}{\partial \psi} V_1 \right\rangle \right) d\varphi - \frac{\nu_1}{\nu_0} V_1 = \tag{2.4}$$
$$= -\left(G' / (8\nu_0)\right)(1 - \cos 2\psi) + \frac{1}{8}\left(H^2 + G^2 + H'/\nu_0\right)\sin 2\psi +$$
$$+ \frac{1}{16} HG(1 - \cos 4\psi) - \frac{1}{32}\left(H^2 - G^2\right)\sin 4\psi$$

etc. We can similarly find the next coefficients ν_i and V_{i-1} of asymptotic expansions (2.2). Note that when constructing the kth approximation, the determination of function V_k from (2.4) is unnecessary, since the error of the calculations due to integration \n the interval $0 \leqslant s \leqslant L\varepsilon^{-1}$ will be $O(\varepsilon^k)$, while taking V_k into account leads to the so-called "improved" k-th approximation [9, 10].

Using expressions (2.3) and (2.4) derived above, the required variables ψ and φ can be written as follows with an absolute accuracy $O(\varepsilon^3)$ for all $0 \leqslant s \leqslant L\varepsilon^{-1}$

$$\psi^{(k)}(s,\varepsilon)=\int_0^s \nu^{(k)}(\varepsilon y,\varepsilon)dy, \quad k=0,1,2,3,\ldots; \quad s=x\varepsilon^{-1}$$

$$\varphi^{(3)}(x,\varepsilon)=\psi^{(3)}(s,\varepsilon)+\varepsilon V_1\left(x,\psi^{(2)}(s,\varepsilon),\varepsilon\right)+\varepsilon^2 V_2\left(x,\psi^{(1)}(s,\varepsilon),\varepsilon\right) \tag{2.5}$$

Here the functions $\nu^{(k)}$ are defined in (2.3). Substitution of the higher-order approximations for the phase ψ into the coefficients $V_{1,2}$ is not justified and, moreover, we can take $V_2(x, \psi^{(1)}, 0)$ and drop terms $O(\varepsilon)$ in $\psi^{(1)}(s, \varepsilon)$. We recall that the dependence of the functions ν_i and V_i on ε is due to the possible non-smooth relation $g(x, \varepsilon)$ in accordance with (1.2); We assume that the Lipschitz condition is satisfied. Note that $V_i(x, \pi n, \varepsilon) = 0$, i.e. the boundary conditions for $\psi^{(k)}$ and $\varphi^{(k)}$ are identical. Using the approximate expressions for ψ and φ obtained, we can calculate the eigenvalues ε_n of the parameter ε with the required accuracy, and also $\omega_n = \varepsilon_n^{-1}$ and $\lambda_n = \omega_n^2$. As pointed out, the quantities ε_n represent small periods of natural oscillations while ω are high frequencies when $n \gg 1$.

Similar expansions are applicable in problem (1.1) for which we have structural representations of the coefficients p and f that are more general than (1.2). For example, we can take functions of the form

$$f\equiv\omega^K r(x,\varepsilon)+\omega^{K-1}q(x,\varepsilon), \quad p\equiv\omega^{K-2}w(x,\varepsilon), \quad K\geqslant 2, \quad \omega=\sqrt{\lambda}=\varepsilon^{-1}$$

Here the standard requirements that the functions r and w should not change in sign must be satisfied.

3. DETERMINATION OF THE NATURAL FREQUENCIES AND PHASES OF THE OSCILLATIONS

The eigenvalues of the parameters ε_n, $\omega_n = \varepsilon_n^{-1}$, $\lambda_n = \omega_n^2$ of problem (1.1), (1.2) are calculated approximately from the final condition for ψ (2.1) and (2.4), (2.5). For convenience we will introduce the new unknown $z = \pi n\varepsilon$, the eigenvalues of which are bounded as $n \to \infty$. With an error $O(\varepsilon_n^4) = O(n^{-4})$ for z we obtain the equation

$$z=\alpha_0+\alpha_1\zeta z+\alpha_2(\zeta z)^2+\alpha_3(\zeta z)^3+O\left((\zeta z)^4\right), \quad \zeta=(\pi n)^{-1}\ll 1$$

$$\alpha_i=\alpha_i(\zeta z)=\int_0^L \nu_i(x,\zeta z)dx, \quad i=0,1,2,3; \quad \varepsilon=\zeta z \tag{3.1}$$

The approximate solution $z_n = z(\zeta)$ with an error $O(\zeta^4)$ is constructed using the asymptotic expansion procedure [11, 13]. Estimates of the absolute and relative errors in determining z_n are identical. The required values of z_n are most conveniently found by successive approximations in powers of the parameter ζ; in particular, when $k \leqslant 3$ we have

$$z_n^{(3)}=\alpha_0+\alpha_1^{(2)}\zeta z_n^{(2)}+\alpha_2^{(1)}\left(\zeta z_n^{(1)}\right)^2+\alpha_3^{(0)}\left(\zeta z_n^{(0)}\right)^3, \quad \left|z_n-z_n^{(3)}\right|=O\left(\zeta^4\right)$$

$$z_n^{(0)}=\alpha_0, \quad z_n^{(1)}=\alpha_0\left(1+\alpha_1(0)\zeta\right), \quad z_n^{(2)}=\alpha_0\left(1+\alpha_1\left(\zeta z_n^{(1)}\right)\zeta\left(1+\alpha_1(0)\zeta\right)\right)+ \tag{3.2}$$

$$+\alpha_2(0)\zeta^2\alpha_0^2\Big), \quad \alpha_1^{(2)}(\zeta)=\alpha_1\left(\zeta z_n^{(2)}(\zeta)\right), \quad \alpha_2^{(1)}(\zeta)=\alpha_2\left(\zeta z_n^{(1)}(\zeta)\right), \quad \alpha_3^{(0)}=\alpha_3(0)$$

The intermediate approximations $z_n^{(k)}$, $k = 0, 1, 2$ differ from $z_n^{(3)}$ by an amount $O(\zeta^{k+1})$, which can be established by simple estimates. Then, using the values of $z_n^{(k)}$ obtained (3.2) one can determine in an elementary way the required eigenvalues of the parameters $\varepsilon_n^{(k)}$, $\omega_n^{(k)}$, $\lambda_n^{(k)}$.

In accordance with (3.1) we have $\varepsilon_n^{(k)} = \zeta z^{(k)}(\zeta) = z_n^{(k)}/(\pi n)$ in particular, for $k = 0, 1, 2, 3$. The parameter $\varepsilon_n^{(k)}$ is found with an absolute error $O(\zeta^{k+2})$; the relative error will be $O(\zeta^{k+1})$, as is the case for $z_n^{(k)}$. We are particularly interested in determining the eigenvalue of the parameter $\omega_n = \varepsilon_n^{-1}$

$$\omega_n^{(k)}=\pi n/z_n^{(k)}, \quad \varepsilon_n^{(k)}=z_n^{(k)}/(\pi n), \quad \lambda_n^{(k)}=\omega_n^{(k)2} \tag{3.3}$$

The quantity $\omega_n^{(k)}$ with absolute error $O(\zeta^k)$ characterizes the value of the natural frequency of the oscillations. When $k = 0$ we obtain the expression $\omega_n^{(0)} = \pi n/\alpha_0$ corresponding to classical formula (1.3), since $\lambda_n^{(k)} = \omega_n^{(k)2}$. Naturally, the absolute error in calculating $\lambda_n^{(k)}$ is generally a quantity $O(\zeta^{k-1})$. The relative error $O(\zeta^{k+1})$ remains as before, as also for the quantities $z_n^{(k)}$, $\varepsilon_n^{(k)}$, $\omega_n^{(k)}$, which is obvious.

The following fundamental result has been obtained. Formulae (3.1)–(3.3) specify in an explicit analytical form the required refined values of ε_n, ω_n, λ_n. They are elementary – they can be reduced to simple quadratures of known functions – and are often determined in analytical form. The formulae can also be used for fairly precise calculations for values of the mode number n that are "not very large". These expressions are obviously suitable with an acceptable accuracy in applied problems when $n \geqslant 1$; for $n = 2, 3, \ldots$ the relative error of calculations does not exceed 10^{-2}–10^{-3}, provided, of course, that the functions p, r and q are sufficiently continuous (see the examples in Section 6 below). In specific problems it is advisable to make a preliminary investigation on the basis of numerical calculations for the lower oscillation modes $n = 1, 2, \ldots$ and to compare the results with the asymptotic expressions (3.1) – (3.3) obtained (example 6.4).

We will briefly describe a scheme for determining the phases $\varphi_n(x)$ of the natural oscillations. From (2.3), (2.5) and (3.3) we obtain expressions for the average frequencies $\nu_n^{(k)}(x)$ and phases $\psi_n^{(k)}(x)$ of the form

$$\nu_n^{(k)}(x) = \sum_{i=0}^{k} \left(\varepsilon_n^{(k)}\right)^i \nu_i\left(x, \varepsilon_n^{(k)}\right) = \sum_{i=0}^{k} \left(\varepsilon_n^{(k-i)}\right)^i \nu_i\left(x, \varepsilon_n^{(k-i)}\right) + O\left(\zeta^{k+1}\right)$$

$$\psi_n^{(k)}(x) = \omega_n^{(k)} \int_0^x \nu_n^{(k)}(y)\,dy, \quad k = 0, 1, 2, 3, \ldots; \quad \zeta = \frac{1}{\pi n}, \quad n \gg 1 \tag{3.4}$$

From the constructions and estimates obtained in Sections 2 and 3 we obtain the following estimates for the rate of convergence of $\nu_n^{(k)}$ and $\psi_n^{(k)}$ to the exact expressions

$$\left|\nu_n(x) - \nu_n^{(k)}(x)\right| \leqslant c\zeta^{k+1}, \quad \left|\psi_n(x) - \psi_n^{(k)}(x)\right| \leqslant c\zeta^k, \quad 0 \leqslant x \leqslant L \tag{3.5}$$

The required phases $\varphi_n(x)$ of the natural oscillations with the required degree of accuracy with respect to ζ are determined from formulae (2.2) and (3.3)

$$\varphi_n^{(k)}(x) = \psi_n^{(k)}(x) + \sum_{i=1}^{k} \left(\varepsilon_n^{(k)}\right)^i V_i\left(x, \psi_n^{(k)}(x), \varepsilon_n^{(k)}\right), \quad \left|\varphi_n - \varphi_n^{(k)}\right| \leqslant c\zeta^k \tag{3.6}$$

In expressions (3.4) and (3.6) we can drop terms $O(\zeta^k)$. We recall that the functions V_i (2.4) were constructed so that $V_i(x, \pi n, \varepsilon) \equiv 0$.

A rigorous proof of the convergence to the exact solution as $k \to \infty$ is problematic even in the case of a system that is analytic with respect to x, since the expansions have an asymptotic character and, generally speaking, diverge as $k \to \infty$. It is more convenient to use the method of successive approximations and the corresponding theorem on the contraction operator [11].

It is well known [9–16], that in the asymptotic procedures of the averaging and accelerated convergence methods, connected with the construction of the changes of variables, an increase in the accuracy of the calculations requires considerable smoothness of the equations or the use of smoothing techniques. This is due to the accumulation of the order of derivatives with respect to the slow variable x in the case under consideration; the function N_+ (2.1) is analytic with respect to φ and has the form of an elementary trigonometric polynomial. Analyticity with respect to x of the right-hand sides, however, does not guarantee the convergence of these procedures as $k \to \infty$ because of "resonance" between the fast variable φ and the slow variable x [14].

By combining the averaging and successive-approximation methods [11] we can construct a procedure for the unlimited refinement of the solution $\varphi(x, \varepsilon)$ of Eq. (2.1), which does not require analyticity. We will describe the corresponding scheme rather briefly and discuss its features. At the initial stage we make the change of variables $\varphi \to \psi$ using the known function $V_1(x, \psi, \varepsilon)$ (2.2). By differentiating with respect to s and by appropriate algebraic conversions we obtain a boundary-value problem of the form

$$d\psi/ds = \nu^{(1)}(x,\varepsilon) + \varepsilon^2 M(x,\psi,\varepsilon), \quad \psi(0) = 0, \quad \psi\left(L\varepsilon^{-1}\right) = \pi n$$

$$\varphi = \psi + \varepsilon V_1(x,\psi,\varepsilon), \quad V_1 \equiv \left(2\nu_0(x)\right)^{-1} F_-(x,\psi,\varepsilon) \tag{3.7}$$

$$M \equiv \left[\frac{1}{\varepsilon}\left(N_+(x,\psi+\varepsilon V_1,\varepsilon) - N_+(x,\psi,\varepsilon)\right) - \left(N_+ - \langle N_+\rangle\right)\langle N_+\rangle\right] \times$$

$$\times\left[1 + \frac{\varepsilon}{\nu_0}\left(N_+ - \langle N_+\rangle\right)\right]^{-1}$$

The function M (3.7) is analytic with respect to ψ in a certain strip for sufficiently small $\varepsilon > 0$ (see [12, 14–16]), and continuous with respect to x. With respect to ε it satisfies the Lipschitz condition, which is used when constructing the solution of the boundary-value problem, i.e. when determining ε_n and ω_n, λ_n. When calculating the function $\psi(x, \varepsilon)$ in the range $0 \leqslant x \leqslant L$ we apply to Cauchy problem (3.7) the method of successive approximations

$$\psi^{(k+1)}(x,\varepsilon) = \psi^{(1)}(x,\varepsilon) + \varepsilon\int_0^x M\left(y,\psi^{(k)}(y,\varepsilon),\varepsilon\right)dy, \quad k = 1,2,\ldots \tag{3.8}$$

The function $\psi^{(1)}(x, \varepsilon)$ in (3.8) is a quadrature of $\nu^{(1)}$ (see (2.5)). The successive approximations (3.8) converge absolutely and uniformly when $\varepsilon > 0$ is sufficiently small to a function $\psi(x, \varepsilon)$ that is continuously differentiable with respect to x, where the difference $\psi - \psi^{(1)} = O(\varepsilon)$, $0 \leqslant x \leqslant L$ and ψ satisfies the Lipschitz condition with respect to ε. This property enables us to utilize the procedure for determining the eigenvalues ε_n and ω_n, λ_n using the method of successive approximations [11, 13], similar to algorithm (3.1)–(3.3). It should be noted that formula (3.8) is extremely cumbersome from the computational point of view, since it requires highly accurate integration of a rapidly oscillating function. The required phases $\varphi_n(x)$ for the expressions $\psi_n(x) = \psi(x, \varepsilon_n)$ are calculated using formula (3.7) for the change of variables.

The procedure of successive changes of variables $\varphi \to \psi$ (3.7) can be continued recurrently, leading to the well-known scheme of the accelerated convergence method [12, 14–16]. However, its attractive property of extremely rapid reduction of the error requires, as in the averaging method, considerable smoothness. Computational difficulties also arise due to the catastrophic complication of obtaining the analytical expressions $M^{(k)}(x, \psi^{(k)}, \varepsilon)$as the order of the iteration $k = 1, 2, \ldots$ ($M^{(1)} \equiv M$) increases. These difficulties are aggravated when the Moser smoothing technique [16] is employed.

For practical calculations the use of the above-mentioned procedure, based on the averaging and asymptotic expansion methods, turns out to be quite effective. Later, we will assume that the quantities ε_n, ω_n, λ_n and the functions $\varphi_n(x)$ are known with the required degree of accuracy with respect to the small parameter $\zeta = (\pi n)^{-1}$. The smallness of the error, as has been established, is ensured either by an increase in n (the number of the mode of oscillation or the resonance zone) or by an increase in the iteration number k, or a combination of these factors.

4. DETERMINATION OF THE AMPLITUDES AND SHAPES OF THE NATURAL OSCILLATIONS

After determining the parameters ε_n and the phases $\varphi_n(x)$, the amplitudes $a_n(x)$ can be found, as mentioned in Section, 2, by elementary quadratures

$$a_n(x) = a_n^0 \exp\left[\int_0^x F_+\left(y,\varphi_n(y),\varepsilon_n\right)dy\right], \quad a_n^0 = \text{const} \tag{4.1}$$

Here a_n^0 are arbitrary constants, chosen from the additional conditions, for example, from the condition for normalization "with a weight" $f_\lambda'(x, \lambda_n)$. However, highly accurate integration of the rapidly oscillation function F_+ in (4.1), just like M in (3.8), is extremely difficult for numerical calculations. This is due to the requirement that the integration step should be small: $\Delta y \ll \varepsilon_n \ll 1$ when $n \gg 1$. It is more effective to use the analytical procedure of the averaging method with subsequent use of numerical methods of integrating regular (continuous) functions [9–11] (see Sections 2 and 3).

We will introduce an unknown smooth variable b, corresponding to a, and we will construct the change of variables, close to identity, and such that the equations for b and ψ do not contain ψ with the required

degree of accuracy with respect to ε. This change of variables for the phase $\varphi \to \psi$ is constructed independently in Section 2 (see (2.1)). The change of variables $a \to b$ depends on ψ and also on x and ε; it has the form

$$a = b(1+\varepsilon U(x,\psi,\varepsilon)), \quad U = U_1(x,\psi,\varepsilon)+\varepsilon U_2+\ldots+\varepsilon^{k-1}U_k+\varepsilon^k\ldots$$
$$db/ds = \varepsilon bd(x,\varepsilon), \quad d = d_1(x,\varepsilon)+\varepsilon d_2+\ldots+\varepsilon^{k-1}d_k+\varepsilon^k\ldots \tag{4.2}$$

The functions U_i and d_i are determined in the same way as V_i and ν_i (2.2)–(2.4) and here the known expressions for V_i and ν_i are used. By means of elementary operations we obtain the required representation; in particular, we obtain the following explicit analytical expressions for $d_{1,2,3}$ and $U_{1,2}$

$$d_1 = \langle F_+\rangle = -\frac{1}{2}h(x),$$
$$d_2 = \langle F_+U_1\rangle + \left\langle \frac{\partial F_+}{\partial\varphi}V_1\right\rangle - d_1\langle U_1\rangle - \left\langle\frac{\partial U_1}{\partial x}\right\rangle = \frac{\nu_0}{8}H(x)G(x,\varepsilon) - \frac{1}{4}G'(x,\varepsilon)$$
$$d_3 = \left\langle F_+U_2 + \frac{\partial F_+}{\partial\varphi}(U_1V_1+V_2) + \frac{1}{2}\frac{\partial^2F_+}{\partial\varphi^2}V_1^2 + d_2U_1 + d_1U_2 - \frac{\partial U_2}{\partial x}\right\rangle \tag{4.3}$$

$$U_1 = \frac{1}{\nu_0}\int_0^\psi (F_+ - \langle F_+\rangle)d\varphi = -\frac{1}{2\nu_0(x)}N_+(x,\psi,\varepsilon)$$
$$U_2 = \frac{1}{\nu_0}\int_0^\psi \left(F_+U_1 + \frac{\partial F_+}{\partial\varphi}V_1 - d_2 - d_1U_1 - \frac{\partial U_1}{\partial x} - \nu_1\frac{\partial U_1}{\partial\varphi}\right)d\varphi$$

Note that the averages of the functions F_+U_1, $(\partial F_+/\partial\varphi)V_1$, $\partial U_1/\partial\varphi$ and $\partial U_2/\partial\varphi$ are identically equal to zero. The subsequent coefficients of the expansions (4.2) are calculated analytically in the same way as (4.3); it is not required to determine U_k for the k-th approximation (see above).

Further, using the formulae of the change of variables (4.2) we obtain the required amplitude $a_n^{(k)}(x)$ in the required approximation (with error $O(\zeta^k)$)

$$a_n^{(k)}(x) = b_n^{(k)}(x)\left(1+\varepsilon_n^{(k)}U^{(k-1)}\left(x,\psi_n^{(k-1)},\varepsilon_n^{(k-1)}\right)\right), \quad U^{(k-1)} = U_1+\varepsilon U_2+\ldots+\varepsilon^{k-2}U_{k-1}$$
$$b_n^{(k)}(x) = a_n^0\exp\left[\int_0^x d^{(k)}\left(y,\varepsilon_n^{(k)}\right)dy\right], \quad d^{(k)} = d_1+\varepsilon d_2+\ldots+\varepsilon^{k-1}d_k \tag{4.4}$$

We now substitute the known function $a_n^{(k)}(x)$ (4.4) and $\varphi_n^{(k)}(x)$ (3.6) into the formulae the change of variables (3.6); we obtain the required expressions for the shapes $u_n(x)$ of the natural oscillations and their derivatives in the k–th approximation

$$u_n^{(k)}(x) = a_n^{(k)}(x)\sin\varphi_n^{(k)}(x), \quad \left|u_n - u_n^{(k)}\right| \leqslant c\zeta^k, \quad 0\leqslant x\leqslant L$$
$$u_n^{(k)\prime}(x) = \omega_n^{(k)}\nu_0(x)a_n^{(k)}(x)\cos\varphi_n^{(k)}(x), \quad \left|u_n' - u_n^{(k)\prime}\right| \leqslant c\zeta^{k-1} \tag{4.5}$$

The functions $u_n^{(k)}(x)$, $u_m^{(k)}(x)$ (4.5) satisfy the approximate condition of orthogonality when $n, m \gg 1$

$$\left(\lambda_n^{(k)}-\lambda_m^{(k)}\right)^{-1}\int_0^L u_n^{(k)}(x)u_m^{(k)}(x)\left(f\left(x,\lambda_n^{(k)}\right)-f\left(x,\lambda_m^{(k)}\right)\right)dx =$$
$$= \delta_{nm} + O\left(\zeta_*^k\right), \quad \zeta_* = \max\left((\pi n)^{-1},(\pi m)^{-1}\right) \tag{4.6}$$

It is assumed in (4.6) than $n \sim m$; here it is taken into account that the quantities $\omega_n^{(k)}$, $\omega_m^{(k)}$ may differ considerably: $\omega_n^{(k)} = \omega_m^{(k)} = O(1)$.

In the second approximation, refining expression (1.3), by an order of magnitude in terms of powers of $\zeta = (\pi n)^{-1}$, for the variables $a_n(x)$ and $u_n(x)$ we obtain the following representations in an explicit analytical form

$$a_n^{(2)}(x) = A_n^0 \mu_0^{1/2}(x)\left(1 + \varepsilon_n^{(0)} I_-(x)\right), \quad I_\pm = \frac{1}{8}\int_0^x h(y)G(y,0)dy \pm \frac{1}{4}\left(G(x,0) - G(0,0)\right)$$
$$u_n^{(2)}(x) = A_n^0 \mu_0^{1/2}(x)\left(\sin\psi_n^{(2)}(x) + \varepsilon_n^{(0)} I_+(x)\sin\psi_n^{(1)}(x)\right), \quad A_n^0 = \text{const} \tag{4.7}$$

In formulae (4.7) we have dropped terms $O(\zeta^2)$. Note that the expressions for the subsequent approximations have an extremely cumbersome form and if necessary can be obtained using a computer algebra program.

For the approximate solution of system of equations (2.1) for specified $\varepsilon = \varepsilon_n$ we used a procedure which combined the method of change of variables (averaging and accelerated convergence) and the method of successive approximations [11], similar to (3.7) and (3.8). However, it leads to extremely complex calculations in the numerical integration of rapidly oscillating functions (see above).

From the applied point of view we are particularly interested in finding the natural frequencies of oscillation accurately. These quantities are less subject to the influence of perturbing factors, which are difficult to take into account (non-linearities and dissipation of different physical kinds, parameter drift etc.). The natural frequencies are important stable characteristics of mechanical oscillatory systems.

5. OTHER TYPES OF BOUNDARY-VALUE PROBLEMS

The scheme described above for determining the frequencies and shapes of higher modes of oscillation can be extended directly to boundary-value problems with other boundary conditions.

1. Suppose that the boundary conditions for Eq. (1.1) correspond to free ends, i.e. $u'(0) = u'(L) = 0$. We then have the following boundary conditions for the phase φ in the change of variables (2.1)

$$\varphi(0) = \pi/2, \quad \varphi\left(L\varepsilon^{-1}\right) = (\pi/2)(2n+1), \quad n = 1,2,\ldots$$

Note that it would be more convenient to seek a solution in the form

$$u = a\cos\varphi, \quad du/ds = -a\nu_0 \sin\varphi \tag{5.1}$$

The boundary conditions then retain the form (2.1), but the equations for a and φ are changed somewhat. Without repeating similar constructions we will derive, in particular, the coefficients of the change of variables of the type (2.2)–(2.4), (4.2) and (4.3) to obtain the second approximation

$$\nu_0 = (r/p)^{1/2}, \quad \nu_1 = -\tfrac{1}{2}g, \quad \nu_2 = \tfrac{1}{4}H' - \tfrac{1}{8}\nu_0\left(H^2 + G^2\right)$$
$$V_1 = \left(2\nu_0\right)^{-1} F_+(x,\psi,\varepsilon), \quad V_1\left(x,(\pi/2)(2n+1),\varepsilon\right) \equiv 0$$
$$d_1 = -\tfrac{1}{2}h, \quad d_2 = \tfrac{1}{4}G' - \tfrac{1}{8}\nu_0 HG \tag{5.2}$$
$$U_1 = -\left(2\nu_0\right)^{-1} N_+(x,\psi,\varepsilon), \quad U_1\left(x,(\pi/2)(2n+1),\varepsilon\right) \equiv 0$$

The eigenvalues of the parameters ε_n, ω_n, λ_n are determined in the same way as described in Section 3. The formulae for calculating the required functions are practically the same. The calculations are carried out in the same way as when one of the ends is clamped while the other is free.

2. We will briefly investigate the asymptotic form of the frequencies and forms of the higher modes of oscillations of system (1.1), (1.2) for the general case of elastic clamping of the ends: $pu' \mp k$, $u = 0$ when $x = 0, L$, where $k_{0,L}$ are the elasticity coefficients concentrated at the ends. We must conclude from physical considerations that when $\omega \to \infty$ the term characterizing the distributed elasticity at $x = 0, L$, i.e. $u'(0) = u(L) = 0$, if $k_{0,L} \to \infty$, is of fundamental importance. Hence, as in the case of free ends (see Section 1) it is more convenient to introduce the change of variables (5.1), which enables us to represent the initial boundary-value problem in the form

$$da/ds = \varepsilon a F_+(x, \varphi - \pi/2, \varepsilon), \quad a(0) = a^0 \neq 0$$
$$d\varphi/ds = \nu_0(x) + \varepsilon N_+(x, \varphi - \pi/2, \varepsilon), \quad x = \varepsilon s \tag{5.3}$$
$$\varphi(0) = \varphi^0 = \operatorname{arctg}(\varepsilon k_0 \mu(0)), \quad \varphi\left(L\varepsilon^{-1}\right) = \varphi^L = \pi n + \operatorname{arctg}\left(\varepsilon k_L \mu_0(L)\right)$$

It follows from (5.3) that the elasticity concentrated at the ends when $k_{0,L} \sim 1$ affects the solution only in the second approximation with respect to ε. We obtain the following expressions for the required coefficients ν_i and V_i

$$\nu_0 = (r/p)^{1/2}, \quad \nu_1 = -\tfrac{1}{2}g, \quad \nu_2 = \tfrac{1}{4}H'\cos 2\varphi^0 - \tfrac{1}{4}G'\sin 2\varphi^0 - \tfrac{1}{8}\nu_0\left(H^2 + G^2\right)$$
$$V_1 = -\tfrac{1}{4}\left(H\left(\cos 2\varphi^0 - \cos 2\psi\right) + G\left(\sin 2\psi - \sin 2\varphi^0\right)\right) \tag{5.4}$$

etc. The quantities ω_n in the principal (zeroth), first and second approximations are determined from conditions of the type (3.1) – (3.3) in the form

$$\varepsilon_n^{(k)} = \frac{1}{\omega_n^{(k)}}, \quad \lambda_n^{(k)} = \omega_n^{(k)2}; \quad \omega_n^{(0)} = \frac{\pi n}{\alpha_0}, \quad \omega_n^{(1)} = \frac{\pi n}{\alpha_0} - \frac{\alpha_1(0)}{\alpha_0}$$
$$\omega_n^{(2)} = \frac{\pi n}{\alpha_0} - \frac{\alpha_1\left(\varepsilon_n^{(0)}\right)}{\alpha_0} - \frac{\alpha_2(0)}{\pi n} + \frac{\delta\varphi^L - \varphi^0}{\alpha_0}, \quad \delta\varphi^L = \varphi^L - \pi n \tag{5.5}$$

It must be borne in mind that, according to (5.3), the quantities $\varphi^0, \delta\varphi^L \sim \zeta$. On the basis of formulae (5.4) and (5.5), similar to (3.4) and (3.6), we find the phases $\varphi_n(x)$, and then the amplitudes $a_n(x)$ and the shapes $u_n(x)$ of the natural oscillations with the required degree of accuracy in ζ (see Section 4).

3. The approach developed above, based on a combination of the averaging and asymptotic expansion methods, can also be extended to generalized boundary-value problems with mixed boundary conditions (for example, periodicity conditions in Hill's problem), conditions which depend on λ and also to more general equations of state, in particular, systems of equations with variable coefficients which depend on λ in a certain fairly arbitrary manner. It is of considerable interest for applications to problems of the theory of elastic and parametric oscillations to develop asymptotic methods of non-linear mechanics for interacting multidimensional systems, and also for partial differential equations, for example, for the asymptotic form of the frequencies and shapes of the natural oscillations of inhomogeneous membranes, including those with a complex boundary.

The use of the approach developed above to construct the frequencies and shapes of higher modes of oscillation with a specified accuracy in combination with the highly effective numerical-analytical method of accelerated convergence [4] to determine the lower frequencies and shapes enables fairly complete investigations of a wide class of mechanical systems to be carried out.

6. ANALYSIS OF SPECIFIC MECHANICAL SYSTEMS

For illustration purposes we will analyse some specific examples which are of interest for applications (the oscillations of inhomogeneous elastic systems and parametric oscillations with complex excitation). We will calculate the higher eigenvalues (the frequencies or parameters) of some oscillatory systems using the procedure proposed in Sections 2, 3 and 5.

6.1. *The longitudinal oscillations of an inhomogeneous rectilinear rod.* The natural frequencies and shapes are determined by solving the Sturm–Liouville problem [1, 5]

$$(ES(x)u')' + \lambda dS(x)u = 0, \quad 0 \leq x \leq l$$
$$1)\ u(0) = u(l) = 0; \quad 2)\ u'(0) = u'(l) = 0 \tag{6.1}$$

Here l is the length of the rod, E is Young's modulus, d is the volume density, $S(x)$ is the cross-sectional area and λ is the constant of separation (the square of the frequency), which is to be determined. To

fix our ideas we will take boundary conditions of the first and second kind in (6.1). The functions u, S, the argument x and the parameter λ will be reduced to dimensionless form so that $E = d = l = 1$. Problem (6.1) will be investigated for $\lambda \gg 1$, i.e. $\lambda dl^2/E \gg 1$ in the initial dimensional variables.

From (1.1) and (1.2) we have $r = p = S(x)$, $q = 0$; reduction to the form of system (2.1) gives $\nu_0 \equiv 1, g \equiv 0, h = S'/S$. Using formulae (2.3) and (5.2) we obtain.

$$\nu_{1,3} \equiv 0, \quad \nu_2^{1,2} = \mp \tfrac{1}{4} h' - \tfrac{1}{8} h^2$$

where the superscripts denote correspondence to boundary conditions 1 and 2 in accordance with (6.1).

Using relations (3.1) – (3.3) and (5.2) we obtain the following required expressions for the natural frequencies with an error $O(\zeta^3)$

$$\omega_n^{1,2} = \pi n + \delta\omega_n^{1,2}, \quad \delta\omega_n^{1,2} = \frac{1}{4\pi n}\left[\pm\frac{S'(x)}{S(x)}\Big|_0^1 + \frac{1}{2}\int_0^1 \left(\frac{S'(x)}{S(x)}\right)^2 dx\right] \tag{6.2}$$

It follows from (6.2) that the relative contribution of the inhomogeneity is $O(\zeta^2)$ while the absolute contribution is $O(\zeta)$; it is due to two factors: local when $x = 0,1$ and integral (quadrature).

Let us consider specific expressions for the function $S(x)$.

Suppose $S(x) = 1 + \theta(2x - 1)$ is a linear function of x, where $|\theta| < 1$; then $S(1/2) = 1$ irrespective of the value of the specified parameter θ while the volume of the rod $V \equiv 1$. Substituting $S(x)$ into (6.2) we obtain the corrections

$$\delta\omega_n^1 = -\tfrac{1}{2}(\pi n)^{-1}\theta^2\left(1-\theta^2\right)^{-1}, \quad \delta\omega_n^2 = \tfrac{3}{2}(\pi n)^{-1}\theta^2\left(1-\theta^2\right)^{-1} \tag{6.3}$$

From an analysis of (6.2) and (6.3) we obtain the following qualitative conclusion: an inhomogeneity that is linear in x in the case of clamped ends reduces the natural frequencies, while for free ends it increases them.

We will now consider the oscillations of a rod of circular cross section, whose radius varies linearly with x: $R(x) = R_0 + \theta(2x - 1)$ with the condition $|\theta| < R_0$. The cross sectional area $S(x) = \pi R^2(x)$, but $S(1/2) = \pi R_0^2$ independently of θ. From (6.2) we obtain the values of the corrections

$$\delta\omega_n^1 = 0, \quad \delta\omega_n^2 = 4(\pi n)^{-1}\theta^2\left(R_0^2 - \theta^2\right)^{-1}, \quad |\theta| < R_0 \tag{6.4}$$

Hence, it follows from (6.4) that in the second approximation with respect to ζ the frequency of longitudinal oscillations of a conical rod is constant with respect to the parameter θ in the case of clamped ends and increases when $\theta \neq 0$ for a rod with free ends. If it is required in this case that the volume of the rod V should be fixed and equal to $\pi\varkappa^2$, where. $\varkappa = \text{const}$, we obtain the following expressions for the required corrections

$$\delta\omega_n^1 = 0, \quad \delta\omega_n^2 = (\pi n)^{-1}\gamma^2\left(1 - \tfrac{1}{3}\gamma^2\right)^{-1}, \quad \gamma^2 = 4\theta^2\varkappa^{-2} < 3 \tag{6.5}$$

Thus, a change in the taper parameter θ for a family of rods of constant volume leads to similar qualitative conclusions regarding the corrections to the natural frequencies of oscillations of the higher modes. Using formulae (6.2) one can calculate the corrections $\delta\omega_n^{1,2}$ for arbitrary differentiable functions $S(x)$. In general they have an arbitrary sign or are equal to zero depending on the effect of the local factor; the integral term is always positive.

6.2. *The vibrations of an inhomogeneous string.* The natural frequencies and forms of transverse vibrations of a tightly stretched string are determined by solving classical boundary-value problems for the equation.

$$Tu'' + \lambda\rho(x)u = 0, \quad \rho(x) = dS(x), \quad 0 \leqslant x \leqslant l \tag{6.6}$$

To fix our ideas we will take as the boundary conditions relations 1 and 2 from (6.1). Here T is the constant tension in the string, $\rho(x)$ is the mass per unit length and l is the length of the string. Without loss of generality we can set $T = l = d = l$. The functions u, ρ, the argument x and the parameter λ are assumed given in dimensionless form.

From (1.1) and (1.2) we have $r \equiv \rho(x), p \equiv 1, q \equiv 0$; transformation of (6.6) to the form (2.1) gives the following expressions: $v_0 = \sqrt{\rho(x)}, g \equiv 0, h = 1/2\ \rho'(x)/\rho(x)$, where $\rho(x) = S(x)$. As in example 6.1, formulae (2.3) give the expressions

$$v_{1,3} \equiv 0, \quad v_2^{1,2} = \mp \tfrac{1}{4} H' - \tfrac{1}{8} v_0 H^2$$

Using relations (3.1)–(3.3) and (5.2) we obtain the natural frequencies $\omega_n^{1,2}$ in the third approximation (the relative error is $O(\zeta^4)$ and the absolute error is $O(\zeta^3)$

$$\omega_n^{1,2} = \pi n / \alpha_0 + \delta\omega_n^{1,2}, \quad \delta\omega_n^{1,2} = -\alpha_2^{1,2}/(\pi n)$$
$$\alpha_0 = \int_0^1 \sqrt{\rho(x)}\,dx, \quad \alpha_2^{1,2} = \mp \frac{1}{4}(H(1) - H(0)) - \frac{1}{8}\int_0^1 v_0(x) H^2(x)\,dx \tag{6.7}$$

We will investigate the behaviour of the correction $\delta\omega_n^{1,2}$ (6.7) for different functions $\rho(x)$. Like (6.2) the integral term is always positive while the local term can have any sign or can be zero. For a specific form of the function $\rho(x)$, i.e. $S(x)$, the quantities α_0, $\alpha_2^{1,2}$ in (6.7) are determined analytically or numerically.

Consider some special cases.

For a linear function $\rho(x) = 1 + \theta(2x - 1)$, $|\theta| < 1$, the expressions for α_0, $\delta\omega_n$ can be found in an analytical form using (6.7)

$$\alpha_0 = 2\left(1 + \theta^2/3\right)\left[(1+\theta)^{3/2} + (1-\theta)^{3/2}\right]^{-1}, \quad \delta\omega_n^1 = -\tfrac{5}{16} D/(\pi n), \quad \delta\omega_n^2 = \tfrac{7}{16} D/(\pi n)$$
$$D(\theta) = 4\theta^2\left(1 + \theta^2/3\right)\left(1 - \theta^2\right)^{-3/2}\left[(1+\theta)^{3/2} + (1-\theta)^{3/2}\right]^{-1} \tag{6.8}$$

The following qualitative result follows from (6.8) analogous to the model of the oscillations of a rod (see (6,3)): a linear change in the density $\rho(x)$ with respect to x leads to a reduction in the frequency of natural vibrations of a string with clamped ends. If the ends of the string are free for transverse displacements, the correction is positive. Note that for this family of strings the volume (mass) is constant: $V(\theta) \equiv 1$ ($m(\theta) \equiv 1$).

Suppose now that the radius of a string of circular cross section varies linearly, while the mass (volume) is fixed and equal to $m(\theta) = \pi$ independently of θ, which leads to the relation $R_0^2 = 1 - \theta^2/3$. Like (6.8) we obtain

$$\alpha_0 = \left(1 - \theta^2/3\right)^{1/2}, \quad \delta\omega_n^1 = -\tfrac{1}{2} D/(\pi n), \quad \delta\omega_n^2 = \tfrac{3}{2} D/(\pi n)$$
$$D(\theta) = 4\theta^2\left(1 - \theta^2/3\right)^{1/2}\left(1 - \tfrac{4}{3}\theta^2\right)^{-2}, \quad \theta^2 < \tfrac{3}{4} \tag{6.9}$$

It follows from (6.9) that $\delta\omega_n^1 < 0$, as was the case of relations (6.3) and (6.8), but unlike (6.4) and (6.5). The corrections to the frequencies in the case of free ends, as above, again turn out to be positive. From relations (6.7) we can determine the principal terms and corrections of the higher vibration frequencies for arbitrary continuous functions $\rho(x)$. Note that the refining corrections can have arbitrary sign due to a considerable influence of the local factor (when $x = 0, 1$).

6.3. *The asymptotic form of the eigenvalues in Hill's problem.* We will consider the problem of highly accurate construction of periodic solutions of the Hill-type equation

$$u'' + [\lambda - q(x)]u = 0, \quad q(x+1) \equiv q(x), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{6.10}$$

Here $q(x)$ is a sufficiently continuous periodic function, which possibly also depends on certain parameters. It is required to obtain the real values of λ_n corresponding to fairly high modes of oscillations $n \gg 1$. An extremely wide class of problems in the theory of oscillations, elasticity theory and mathematical physics can be reduced to relations of the form (6.10) (see, for example, [1, 2, 9, 17]).

We will investigate the situation, often encountered in practice, when $q(x) \equiv q(-x)$ is an even function. Problem (6.10) is then equivalent to two Sturm–Liouville problems with boundary conditions of the first and second kind, to which odd and even solutions correspond. Note that, in addition to $1/n$ resonances, there are $2/n$ resonances, including the first ones for even n. Using the notation in (1.1) and (1.2) we have the following classical problem: the perturbation is of order ε^2, where $\varepsilon = \omega^{-1}$, $\lambda = \omega^2$. Carrying out elementary calculations, similar to those in Section 2, we obtain the required expressions for the functions v_i corresponding to the fifth approximation in powers of ε (the relative error is $O(\varepsilon^6)$), for boundary conditions of the first kind (the plus sign in front of the functions $q'(x)$) and of the second kind (the minus sign)

$$v_0 = 1, \quad v_2 = -\tfrac{1}{2}q(x), \quad v_4 = \tfrac{1}{8}\left(\pm q''(x) - q^2(x)\right), \quad v_{1,3,5} \equiv 0$$
$$v^{(5)}(x,\varepsilon) = v_0 + \varepsilon^2 v_2(x) + \varepsilon^4 v_4(x) \tag{6.11}$$

The natural frequencies can be calculated, with absolute error $O(\zeta^5)$ from expressions (6.11) using the procedure described in Section 3. As a result we obtain the required representations

$$\omega_n^{(5)} = \pi n - \frac{\alpha_2}{\pi n} - \frac{\alpha_2^2 + \alpha_4}{(\pi n)^3}, \quad \alpha_{2,4} = \int_0^1 v_{2,4}(x)\,dx \tag{6.12}$$

It follows from relations (6.11) and (6.12) that there appears to be no difference between the asymptotic forms of the eigenvalues ω_n for conditions of the first and second kind in terms $O(\zeta^3)$, although in v_4 the signs in front of the functions q'' are different. This is due to the fact that, because of the smoothness and periodicity of the function q, the integral of $q''(x)$ is zero. A difference occurs when higher-order terms are taken into account. Further, if the function has zero mean, then $\alpha_2 = 0$ and the correction is positive

$$\delta\omega_n^{1,2} = \tfrac{1}{8}\langle q^2 \rangle (\pi n)^{-3}, \quad \langle q \rangle = 0$$

where $\langle q^2 \rangle$ is the mean of q^2 (see Mathieu's equation etc. [2, 9, 17, 17]). In the general case, when the $q(x)$ is not even, a special procedure can be developed for the asymptotic form of the eigenvalues λ_n of the periodic boundary-value problem. For discontinuous functions $q(x)$ (Meisner's equation [17]) using (6.11) we can write the required approximations up to terms $O(\zeta^3)$.

6.4. *Spatial oscillations of an artificial satellite.* We will assume that the polar axis of a dynamically symmetrical artificial satellite undergoes oscillations of arbitrary amplitude (or rotation) in the plane of a circular orbit. A small spatial perturbation of these motions then occurs and small angular deviations of the axis of the satellite from the plane of the orbit will be investigated. The corresponding generalized problem for determining the periodic solutions in dimensionless form is described by the following expressions [18] (to fix our ideas we will consider the case of an oblate body)

$$u'' + 16\mathbf{K}^2(k)\left[(\chi + k\,\mathrm{cn}\,\theta)^2 - k^2\,\mathrm{sn}^2\,\theta\right]u = 0 \tag{6.13}$$
$$u = u(x), \quad 0 \leqslant x \leqslant 1, \quad u(0) = u(1), \quad u'(0) = u'(1)$$
$$k^2 = 2h_0\chi^2, \quad \chi = (3(\alpha - 1))^{-1/2}, \quad 0 \leqslant k < 1, \quad 1 < \alpha \leqslant 2$$

Here u is the angular deviation of the polar axis, the prime denotes differentiation with respect to the argument $x = w/(2\pi)$, w is the independent variable, characterizing plane oscillations in the "action-angle" variables, $\theta = 4\mathbf{K}(k)x$ is the argument of the Jacobi elliptic functions cn and sn and $\mathbf{K}(k)$ is the complete elliptic integral of the first kind. The equation contains two constant parameters: the modulus k and the parameter χ; h_0 is the dimensionless energy of plane oscillations and α is the ratio of the polar and equatorial moments of inertia.

In the case of a prolate body ($0 \leqslant \alpha < 1$) the equation of small oscillations of the type (6.13) is modified [18].

Because of the evenness with respect to x of the coefficient of u, periodic problem (6.13) is equivalent to two Strum-Liouville problems (see example 6.3). The asymptotic form of the eigenvalues $\chi_n(k)$ is constructed for sufficiently large values of n, which determine the number of the resonance zone, and

for $0 \leqslant k < 1$. Using the notation of (1.1) and (1.2) we have

$$\lambda = \omega^2 = (4\mathbf{K}\chi)^2, \quad \varepsilon = \omega^{-1}, \quad \varkappa = 4\mathbf{K}k$$
$$p = r \equiv 1, \quad q = -2\varkappa \operatorname{cn}\theta - \varepsilon\varkappa^2\left(\operatorname{cn}^2\theta - \operatorname{sn}^2\theta\right) \tag{6.14}$$

We will determine the eigenvalues $\omega_n(\varkappa)$ of the problem presented in (6.14) in the second approximation with respect to the parameter $\zeta = (\pi n)^{-1}$, i.e. with a relative error $O(\zeta^3)$ for $\varkappa \sim 1$, when the value of k is "not very close" to $k = 1$. In this approximation the quantities $\omega_n^{1,2}(\varkappa)$ for both boundary-value problems are the same. In accordance with the constructions in Sections 2 and 5, taking expressions (6.14) into account we obtain

$$\nu_0 = 1, \quad \nu_1 = \varkappa \operatorname{cn}\theta + \tfrac{1}{2}\varepsilon\varkappa^2\left(\operatorname{cn}^2\theta - \operatorname{sn}^2\theta\right), \quad \nu_2 = -\tfrac{1}{2}\varkappa^2 \operatorname{cn}^2\theta$$
$$\omega_n(\varkappa) = \pi n + \frac{1}{2\pi n}\varkappa^2\left\langle \operatorname{sn}^2\theta\right\rangle, \quad \left\langle \operatorname{sn}^2\theta\right\rangle = \frac{1}{k^2}\left(1 - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right)$$
$$\chi_n(k) = \frac{\pi n}{4\mathbf{K}(k)} + \frac{2}{\pi n}(\mathbf{K}(k) - \mathbf{E}(k)) \tag{6.15}$$

Here $\mathbf{E}$ is the complete elliptic integral of the second kind.

The following estimate follows from (6.15)

$$\chi_n \approx (n/2)\left(1 - k^2/4\right)$$

when $k4 \ll 1$, and as $k \to 1$ the principal term of the expression for χ_n decreases monotonically, since $\mathbf{K} \to \infty$.

Numerical calculations for $\chi_n^{1,2}$ with $0 \leqslant k \leqslant 0.999$ are shown in the figure (the superscript 1 corresponds to odd oscillations and the superscript 2 corresponds to even oscillations). They are obtained by the highly accurate accelerated-convergence method [4] and confirm the acceptable accuracy of expressions (6.15) when $n = 2$ if $0 \leqslant k \leqslant 0.2$; the curves of χ_2^1 and χ_2^2 diverge somewhat. For $n = 3$ there is good agreement when $0 \leqslant k \leqslant 0.4$; the divergence is considerably less. An increase in n of the resonance zone leads to an increase in accuracy and an extension of the range of values of k. The curves of $\chi_n^1(k)$ and $\chi_n^2(k)$ diverge much less and bend in accordance with the asymptotic form of the principal term in (6.15). The following qualitative result is of interest

$$\chi_n^{1,2}(k) \to 1, \quad \alpha_n^{1,2}(k) \to 4/3, \quad k \to 1, \quad n > 2$$

Note that the case of prolate body and rotational modes were investigated similarly using appropriate modified equations in [18].

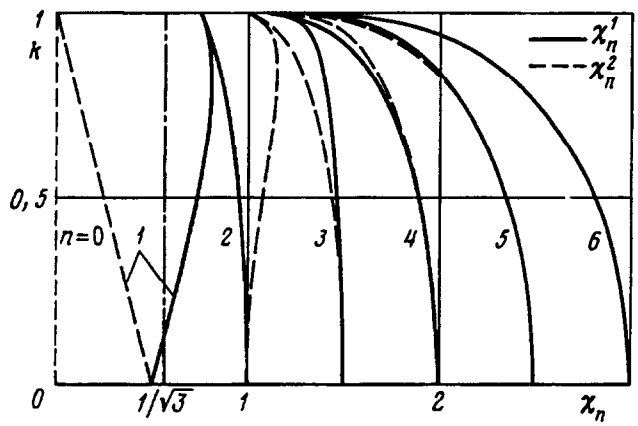

Fig. 1

Hence, the calculation of interesting specific problems confirms the high efficiency of the asymptotic approach developed when investigating higher modes of oscillations of mechanical systems.

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